Half-period delayed feedback control for dynamical systems with symmetries

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Delayed feedback control (DFC), proposed by Pyragas [Phys. Lett. A **170**, 421 (1992)], is a simple and practical method of controlling chaos in continuous dynamical systems. However, it had been proved that the DFC has a limitation; that is, any hyperbolic unstable periodic orbit (UPO) with an odd number of real characteristic multipliers greater than unity can never be stabilized by the DFC. In this paper, to overcome this limitation, we propose a modified DFC, ''half-period delayed feedback,'' of which the delay time is a half of the period of the UPO. We apply it to stabilizing self-symmetric directly unstable periodic orbits of the Duffing equation. This modified DFC can also be generalized to a form stabilizing symmetric periodic orbits in some systems with symmetries such as the Lorenz equation. [S1063-651X(98)08508-0]

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I. INTRODUCTION

Delayed feedback control (DFC), proposed by Pyragas [1], is a method of controlling chaos in continuous dynamical systems. The DFC is based on the feedback of the difference between the current state of a system and a state delayed by the period of a desired unstable periodic orbit (UPO). This is a convenient method that does not require a preliminary calculation of the UPO. Because of this convenience, the DFC method has been applied to controlling chaos in some real systems such as a laser system [2], electronic systems [3,4], and a magnetoelastic system [5].

However, the stability analysis required for determining the feedback gain is very difficult, because the controlled system is described by a delay-differential equation, the state space of which is infinite dimensional. Moreover, it has been proved that the DFC has a limitation, that is, any hyperbolic UPOs with an odd number of real characteristic multipliers greater than unity can never be stabilized by the DFC.

This limitation of the DFC was first proved by Ushio [6] for discrete time systems; then it was proved that the DFC for continuous time systems also has the same limitation [7]. Just *et al.* [8] discussed basically the same result. Pyragas [9] showed the same limitation for stabilizing equilibrium points in two-dimensional continuous systems, and Konishi and Kokame [10] proved this limitation for two-dimensional discrete systems. It was also proved that every control method in which the feedback term vanishes when an orbit of the same period as the delay time is stabilized has the same limitation [11]. Hence it seems hard to overcome this limitation by any modifications of the DFC proposed so far, including the extended time delay autosynchronization (ETDAS) [9,12,13].

In order to overcome this limitation, we propose a modification of the DFC, ''half-period delayed feedback control,'' for stabilizing self-symmetric directly unstable periodic orbits of the Duffing equation. We also show that a symmetric periodic orbit of the Lorenz equation can be stabilized by a more generalized form of this modified DFC.

The Duffing equation is a two-dimensional nonautonomous differential equation with periodic forcing [14]. Since the stroboscopic map of this equation is area contracting, a UPO of it is either directly unstable or inversely unstable. A directly unstable orbit has one characteristic multiplier greater than unity. Thus directly unstable orbits can never be stabilized by the DFC due to the above limitation. Our purpose is to propose a method for stabilizing UPOs of this type.

On the other hand, the Duffing equation has a symmetry: if $\mathbf{x}(t)$ is a solution of the equation, then $-\mathbf{x}(t-T/2)$ is also a solution of it. Here $T(=2\pi)$ is the period of the external forcing. This is due to the fact that the equation is composed of a polynomial including only odd-order terms, and the external forcing is $B\cos t$, which has a symmetry $B\cos t=-B\cos(t+T/2)$. In particular, we call a solution satisfying $\mathbf{x}(t)=-\mathbf{x}(t-T/2)$ a self-symmetric solution. It is easy to see that a self-symmetric solution is T periodic.

The most important property of the Duffing equation we utilize in this study is that a directly unstable orbit is also self-symmetric in many cases. On the basis of this property, we propose a modification of the DFC for stabilizing self-symmetric directly unstable orbits in the Duffing equation, and confirm its feasibility by numerical experiments. We also generalize it to control other systems with similar symmetries such as the Lorenz equation.

This paper is organized as follows. Section II briefly introduces the DFC and explains its limitation. Section III describes some properties, in particular, a symmetry of the UPOs of the Duffing equation. Section IV proposes a half-period DFC for stabilizing self-symmetric directly unstable periodic orbits in the Duffing equation, which cannot be stabilized by the original DFC. Section V generalizes this modified DFC to a form controlling chaos in more general equations with symmetry, in particular the Lorenz equation.

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Section VI shows some numerical experiments on the stabilization of UPOs in the Duffing equation and the Lorenz equation using the half-period DFC. Section VII presents our conclusion.

II. DFC AND ITS LIMITATION

Let us consider the stabilization of a UPO contained in a chaotic attractor of an *n*-dimensional ordinary differential equation

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t),t)\quad(\mathbf{x}\in\mathbb{R}^n),\tag{1}$$

by the DFC [1],

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t),t)+K(\mathbf{x}(t-T)-\mathbf{x}(t)).\tag{2}$$

Here K is an $n\times n$ feedback gain matrix, and $T(>0)$ is the period of the target UPO. When a solution of Eq. (2) converges to a T-periodic orbit, the feedback term $K(\mathbf{x}(t-T)-\mathbf{x}(t))$ vanishes, so this is guaranteed to be a solution of the uncontrolled system (1), that is, the target UPO is stabilized.

The DFC is a simple and convenient method which does not need preliminary calculation of the UPO, and so it is applicable to controlling unknown systems. However, since the controlled system (2) is described by a differential-difference equation with the delay time T, the determination of the feedback gain K is very difficult even if the function of the system, f, is known. In addition to the above difficulty of the stability analysis, recently it has been shown that the DFC has a limitation expressed by the following theorem.

Theorem. Suppose that Eq. (1) is a nonautonomous system that is T periodic with respect to time t. If the number of real characteristic multipliers of a hyperbolic UPO greater than unity is odd, then the UPO cannot be stabilized by the DFC.

This theorem was first proved by Ushio about the DFC for controlling discrete time systems [6], and then extended to the case of continuous systems [7]. This limitation holds for every control method with which the feedback term vanishes when a periodic orbit of a certain period is stabilized. Therefore some extended DFCs such as the ETDAS [9,12,13], or their application to controlling autonomous systems, also have this limitation [11].

III. UPOS OF THE DUFFING EQUATION AND THEIR SYMMETRY

The Duffing equation is a two-dimensional nonautonomous differential equation [14],

$$\dot{x}=y,\quad \dot{y}=-\delta y-\alpha x-x^3+B\cos t.\tag{3}$$

This is an oscillatory system with a sinusoidal periodic forcing of the period $T=2\pi$. So the period of every periodic solution is an integer times of 2π. In what follows, we deal with 2π-periodic solutions.

Since the damping coefficient δ is positive in general, the stroboscopic map, which is defined by a 2π-periodic plotting of the trajectory of solutions, is area contracting on the (x,y) plane. Therefore, a completely unstable periodic orbit, both of whose characteristic multipliers are greater than unity in their magnitudes, does not exist. On the other hand, the product of the characteristic multipliers of any orbit is positive because the stroboscopic map is orientation preserving. Therefore, characteristic multipliers, denoted by λ_1 and λ_2, of any UPO of the Duffing equation (3) satisfy $0<\lambda_2<1<\lambda_1$ or $\lambda_2<-1<\lambda_1<0$. A UPO satisfying the former is called a directly unstable orbit, and one satisfying the latter is called an inversely unstable orbit. Directly unstable periodic orbits cannot be stabilized by the DFC due to the above mentioned limitation.

On the other hand, the Duffing equation (3) has a kind of symmetry; that is, when $(x(t),y(t))$ is a solution of it, $(-x(t-\pi),-y(t-\pi))$ is also a solution. We can easily prove this fact by noticing that the right hand side of Eq. (3) contains only terms of odd number order with respect to x and y, and that the external forcing has a symmetry $B\cos t=-B\cos(t+\pi)$. (When the external forcing contains a constant term, such as $B\cos t+B_0$, however, this symmetry does not exist.)

We call a solution satisfying $(x(t),y(t))=(-x(t-\pi),-y(t-\pi))$ at every time t a "self-symmetric solution." A self-symmetric solution is 2π periodic. This is easily shown as follows: If $(x(t),y(t))$ is self-symmetric, then

$$\begin{aligned}(x(t-2\pi),y(t-2\pi))&=(x(t-\pi-\pi),y(t-\pi-\pi))\\&=(-x(t-\pi),-y(t-\pi))=(x(t),y(t)).\end{aligned}$$

The most important property of the Duffing equation we utilize in this study is that a directly unstable solution is also self-symmetric in many cases. This does not always hold; however, its converse, "a self-symmetric unstable solution is directly unstable," is true, as shown in the Appendix.

Using the symmetry that many directly unstable orbits possess, in Sec. IV we will propose a modification of the DFC to stabilizing self-symmetric directly unstable periodic orbits of the Duffing equation.

IV. HALF-PERIOD DELAYED FEEDBACK

To stabilize self-symmetric directly unstable periodic orbits, we propose a modified delayed feedback control,

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t),t)-K(\mathbf{x}(t-T/2)+\mathbf{x}(t)).\tag{4}$$

Here $\mathbf{x}=(x,y)$. The function f denotes the right hand side of the Duffing equation (3), and K is a 2×2 gain matrix. The delay time of the feedback is set to $T/2=\pi$, a half of the period of the target directly unstable periodic orbit, instead of $T=2\pi$ as for the original DFC. Hence the feedback term does not necessarily vanish when a 2π-periodic UPO is stabilized, but it vanishes only when a self-symmetric orbit is stabilized. Therefore, the control described by Eq. (4) is expected to stabilize self-symmetric directly unstable orbits selectively. We will call this control half-period delayed feedback control.

Determination of the value of the gain K of this feedback is difficult, as is the case for the original DFC. However, the limitation that directly unstable orbits cannot be stabilized is expected to be dissolved for the following reason. As shown in Ref. [7], since a 2π-periodic orbit is not created, and does not vanish nor change its location, when the gain K is varied in the original DFC, the pitchfork and transcritical bifurcations cannot occur, which is inevitable if a directly unstable orbit is to be stabilized. For the half-period DFC, however, the feedback term vanishes only when a self-symmetric orbit is stabilized. Thus such a restriction of bifurcations no longer exists.

However, this only implies such a limitation does not explicitly hold for the half-period DFC; thus it is not proved, of course, that a directly unstable orbit can always be stabilized. Since the stability analysis for the determination of the gain K is as difficult as for the original DFC, in Sec. VI we will discuss the ability of the proposed DFC to stabilize self-symmetric directly unstable orbits by numerical calculations.

V. GENERALIZATION OF THE HALF-PERIOD DFC

The half-period DFC [Eq. (4)] proposed in Sec. IV is intended to control the Duffing equation (3). However, in general it is applicable to any n-dimensional system described by Eq. (1) when the function f in the right hand side of Eq. (1) is expanded with respect to time t as follows:

$$f(\mathbf{x}(t),t)=f_0(\mathbf{x}(t))+\sum_{k=0}^{\infty} g_k(\mathbf{x}(t))\cos\frac{2\pi(2k+1)}{T}t + \sum_{k=0}^{\infty} h_k(\mathbf{x}(t))\sin\frac{2\pi(2k+1)}{T}t. \tag{5}$$

Here we assume that $f_0(\mathbf{x}(t))$ is a polynomial of an odd order with respect to the components of $\mathbf{x}$, while $g_k(\mathbf{x}(t))$ and $h_k(\mathbf{x}(t))$ are polynomials of an even order of $\mathbf{x}$. It is easy to see that Eq. (1) with these conditions has the same symmetry as the Duffing equation (3), as shown in Sec. III.

The half-period DFC can also be extended to the form controlling systems with more general symmetries as follows [15]: Here we consider nonautonomous systems described by Eq. (1). We assume the function f in Eq. (1) is T periodic with respect to t, and has the symmetry,

$$f(S\mathbf{x},t+T/2)=Sf(\mathbf{x},t), \tag{6}$$

$$S^2=I_n. \tag{7}$$

Here S is an $n\times n$ matrix, and I_n is the n-dimensional unit matrix. It is easy to see that if $\mathbf{x}(t)$ is a solution of Eq. (1) then $S\mathbf{x}(t-T/2)$ is also a solution. We call a solution that satisfies $S\mathbf{x}(t-T/2)=\mathbf{x}(t)$ at every time t a self-symmetric solution. It is also clear that a self-symmetric solution is T periodic. Therefore, such an orbit is expected to be stabilized by a delayed feedback,

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t),t)+K(S\mathbf{x}(t-T/2)-\mathbf{x}(t)). \tag{8}$$

This is a natural generalization of the half-period DFC [Eq. (4)]. The DFC [Eq. (4)] for the Duffing equation can be regarded as a special case of Eq. (8) with the dimension $n=2$ and $S=-I_2$. System (1), with a function f defined by Eq. (5), is also a special case with an $n\times n$ matrix $S=-I_n$.

In Eq. (8), the feedback term is based on the state variable $\mathbf{x}$ itself. However, it can easily be generalized to the form using general linear output, $C\mathbf{x}$, of the sysmem as $K(CS\mathbf{x}(t-T/2)-C\mathbf{x}(t))$. Here C is an $l\times n$ output matrix, and K is an $n\times l$ gain matrix.

Although system (1) considered above is a nonautonomous system with a T-periodic function, the generalized half-period DFC can also be applied to controlling autonomous systems. When Eq. (1) is an autonomous system, f does not depend on time t explicitly. Hence the symmetry is expressed by $f(S\mathbf{x})=Sf(\mathbf{x})$, and if $\mathbf{x}(t)$ is a solution then $S\mathbf{x}(t)$ is also a solution. We may call a T-periodic orbit (T is an arbitrary positive number) that satisfies $S\mathbf{x}(t-T/2)=\mathbf{x}(t)$ self-symmetric. Such a periodic orbit may be stabilized by feedback [Eq. (8)].

As an example of an autonomous system with such a symmetry, here we deal with the Lorenz equation

$$\begin{aligned}\dot{x}&=-\sigma(x-y),\\ \dot{y}&=rx-y-xz,\\ \dot{z}&=xy-bz.\end{aligned} \tag{9}$$

This three-dimensional equation has a symmetry represented by a matrix $S=\mathrm{diag}(-1,-1,1)$. That is, when $(x(t),y(t),z(t))$ is a solution of it, $(-x(t),-y(t),z(t))$ is also a solution. Therefore a self-symmetric solution, which satisfies $x(t)=-x(t-T/2)$, $y(t)=-y(t-T/2)$, and $z(t)=z(t-T/2)$, is a T-periodic solution, and it may be stabilized by the following delayed feedback:

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t))-K\begin{pmatrix} x(t-T/2)+x(t)\\ y(t-T/2)+y(t)\\ z(t)-z(t-T/2)\end{pmatrix}. \tag{10}$$

Here $\mathbf{x}=(x,y,z)^T$, $f(\mathbf{x})$ denotes the right hand side of the Lorenz equation (9), and K is a 3×3 gain matrix. Hence all the feedback terms vanish when a self-symmetric T-periodic orbit is a solution; that is, this control is expected to stabilize such a UPO specifically. This is a generalization of the half-period DFC for autonomous systems, and will be discussed in Sec. VI.

The symmetry described by Eqs. (6) and (7) can be extended to more general forms as

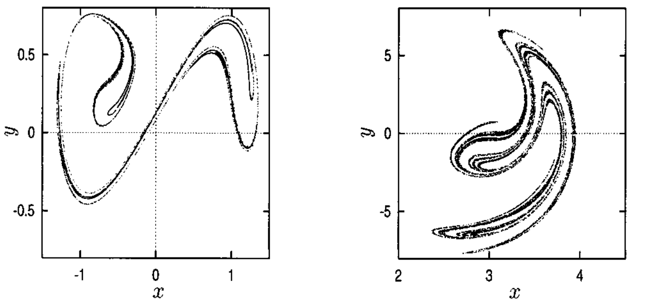
$$f(S\mathbf{x},t+T/m)=Sf(\mathbf{x},t), \tag{11}$$

$$S^m=I_n, \tag{12}$$

where m is an integer satisfying $m\geqslant 2$. We can easily see that if $\mathbf{x}(t)$ is a solution, then $S\mathbf{x}(t-T/m)$ is also a solution. We can consider a delayed feedback control,

$$\dot{\mathbf{x}}(t)=f(\mathbf{x}(t),t)+K(S\mathbf{x}(t-T/m)-\mathbf{x}(t)), \tag{13}$$

to stabilize a self-symmetric and T-periodic orbit that satisfies $S\mathbf{x}(t-T/m)=\mathbf{x}(t)$. This feedback may be called a $1/m$-period DFC. In particular, we consider that a $\frac{1}{3}$-period DFC is very important for controlling chaos in three-phase circuits in electric power systems [16].



(a) $\delta = 0.25, B = 0.3, \alpha = -1.0$ (b) $\delta = 0.1, B = 12.0, \alpha = 0.0$

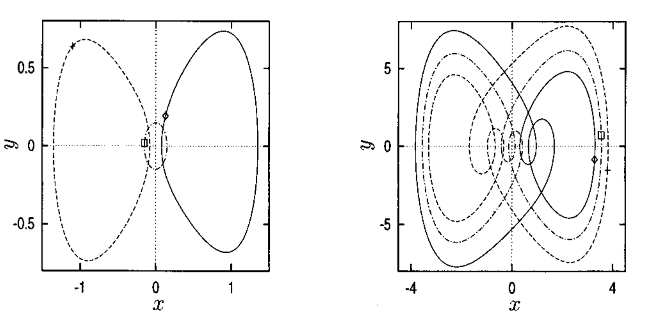
FIG. 1. Chaotic attractors of the Duffing equation (3). Stroboscopic points at $t=2n\pi$. (a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$. (b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$.

VI. NUMERICAL EXPERIMENTS

A. Control of the Duffing system

We made numerical experiments of controlling chaos in the Duffing equation (3) with the following two parameter settings: (a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$; and (b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$. Chaotic attractors shown in Figs. 1(a) and 1(b) are observed at the above parameter settings (a) and (b), respectively (the attractors are expressed as those for the stroboscopic map) [14,17]. In these attractors, the 2π-periodic UPOs shown in Figs. 2(a) and 2(b) are contained. The locations and characteristic multipliers of these UPOs are shown in Table I. For both parameter settings (a) and (b), the UPOs (i) and (ii) are symmetric with each other and inversely unstable, while the UPO (iii) is a self-symmetric directly unstable periodic orbit. Therefore UPOs (i) and (ii) are not limited by the limitation described in Sec. II. In fact, it has been confirmed that they can be stabilized by the original DFC [18]. Moreover, the Duffing equation (3) of setting (a) describes a model of a magnetoelastic oscillator with a chaotic attractor, whose UPOs (i) and (ii) are proved to be stabilized by the DFC in some experiments [5].

Here we try to stabilize the directly unstable periodic orbit (iii), by the DFC [Eq. (4)] with a diagonal feedback gain matrix $K=\mathrm{diag}(k_1,k_2)$. That is, we apply a delayed feedback as follows:



(a) $\delta = 0.25, B = 0.3, \alpha = -1.0$ (b) $\delta = 0.1, B = 12.0, \alpha = 0.0$

◇ : UPO (i) + : UPO (ii) □ : UPO (iii)

FIG. 2. UPOs in chaotic attractors of the Duffing equation (3). (a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$. (b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$.

TABLE I. UPOs of the Duffing equation (3). (x,y) is the fixed point of the stroboscopic map corresponding to each UPO, and (λ_1,λ_2) denotes characteristic multipliers of the fixed point. (a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$. (b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$.

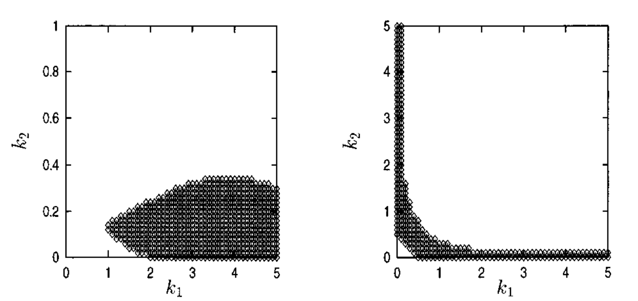
	x	y	λ_1	λ_2
(a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$				
(i)	0.129	0.194	−0.0257	−8.076
(ii)	−1.102	0.636	−0.0257	−8.076
(iii)	−0.149	0.0189	230.49	0.0009
(b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$				
(i)	3.288	−0.853	−0.144	−3.709
(ii)	3.797	−1.535	−0.144	−3.709
(iii)	3.548	0.726	15.326	0.0348

$$\dot{x}(t)=y(t)-k_1(x(t-\pi)+x(t)),$$
$$\dot{y}(t)=-\delta y(t)-\alpha x(t)-x(t)^3+B\cos t-k_2(y(t-\pi)+y(t)). \tag{14}$$

We survey the gain parameters k_1 and k_2 with which the target UPO is stabilized, in the region $0.0\leqslant k_1\leqslant 5.0$ and $0.0\leqslant k_2\leqslant 1.0$ for parameter setting (a), and in the region $0.0\leqslant k_1\leqslant 5.0$ and $0.0\leqslant k_2\leqslant 5.0$ for setting (b). The stability region of UPO (iii) is denoted by marking ◇ in Figs. 3(a) and 3(b). The stability is judged from the numerically calculated maximum characteristic exponent. The result proves that the proposed method is applicable to stabilizing self-symmetric directly unstable periodic orbits. Examples of the process of converging the stroboscopic point to the target UPO under the half-period DFC are displayed in Fig. 4.

The characteristic multipliers shown in Table I imply that UPO (iii) has strong instability in both settings (a) and (b). In particular, two multipliers have extremely different values in (a). Hence it should also be remarkable that such orbits, which are hard to stabilize intrinsically, could be stabilized by this method.

In the parameter region where UPO (iii) cannot be stabilized, the controlled orbits are chaos or 2π-periodic orbits different from each of (i), (ii), and (iii). When a 2π-periodic



(a) $\delta = 0.25, B = 0.3, \alpha = -1.0$ (b) $\delta = 0.1, B = 12.0, \alpha = 0.0$

FIG. 3. Stability region in the k_1-k_2 plane for UPO (iii) of the Duffing equation (3). The stability region is denoted by ◇. (a) $\delta=0.25$, $B=0.3$, and $\alpha=-1.0$. (b) $\delta=0.1$, $B=12.0$, and $\alpha=0.0$.

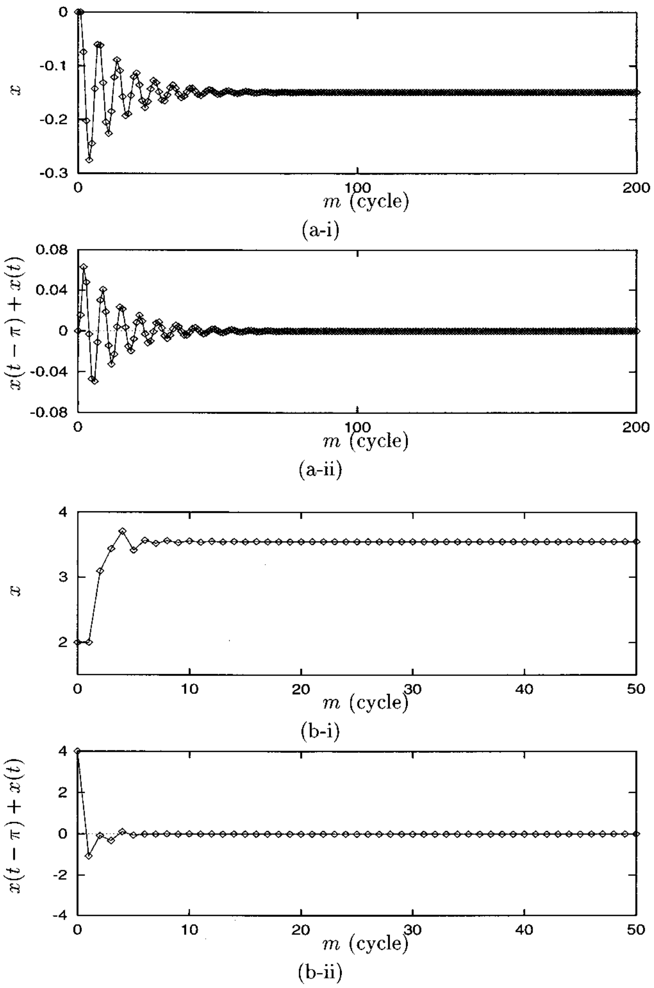


FIG. 4. Examples of the stabilization of UPOs of the Duffing equation (3) by the half-period DFC. (a-i) stroboscopic points and (a-ii) the feedback term $x(t-\pi)+x(t)$ for $\delta=0.25$, $B=0.3$, $\alpha=-1.0$, $k_1=5.0$ and $k_2=0.0$. (b-i) stroboscopic points and (b-ii) the feedback term $x(t-\pi)+x(t)$ for $\delta=0.1$, $B=12.0$, $\alpha=0.0$, $k_1=0.0$, and $k_2=1.0$.

orbit is stabilized, the feedback term does not vanish. Hence it is not a solution of the uncontrolled Duffing equation (3).

Figure 5 is a bifurcation diagram of the stabilization process of UPO (iii) by the half-period DFC [Eq. (14)] with parameter k_1 varied from 0.0 to 1.0, and k_2 fixed to 0.0. In this diagram, two bifurcation processes from different initial values are superimposed. So the bifurcation at the stabilization point is not period doubling, as it seems, but pitchfork. That is, branches of two stable 2π-periodic orbits and UPO (iii) coalesce into a stable orbit near $k_1=0.45$. Such a bifurcation with a coalescence of 2π-periodic orbits can never occur when the original DFC is applied because of the limitation explained in Sec. II [7]. However, for the half-period DFC, the feedback term vanishes only when a self-symmetric orbit is a solution. Thus such a coalescence or disappearance of 2π-periodic orbits can occur in this case.

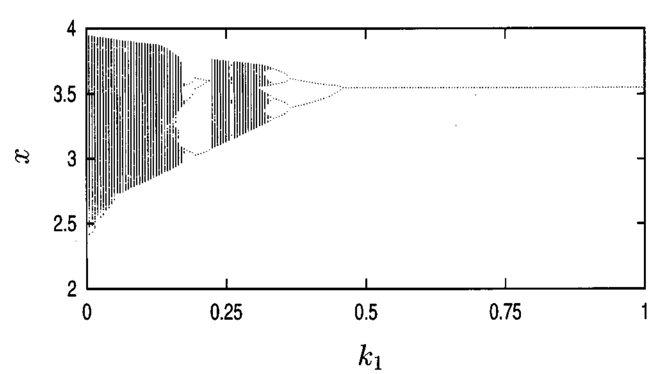


FIG. 5. Bifurcation diagram of attractors for the Duffing equation (3) under the half-period DFC. The parameters settings are $\delta=0.1$, $B=12.0$, $\alpha=0.0$, and $k_2=0.0$.

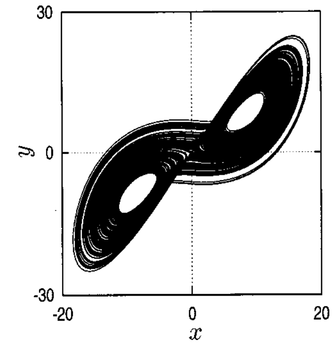


FIG. 6. Lorenz attractor: A chaotic attractor of the Lorenz equation (6) with parameter settings $\sigma=10.0$, $r=28.0$, and $b=8/3$.

B. Control of the Lorenz system

The Lorenz equation (9) has a chaotic attractor shown in Fig. 6 by the system parameters $\sigma=10.0$, $r=28.0$, and $b=\frac{8}{3}$. Among many UPOs contained in this attractor, we choose the self-symmetric one shown in Fig. 7, which satisfies $x(t)=-x(t-T/2)$ and $y(t)=-y(t-T/2)$, with period T as is defined in Sec. V. The period of the UPO is $T\approx 1.559$, and the characteristic multipliers of it are $(\lambda_1,\lambda_2,\lambda_3)=(4.71,\ 1.00,\ 1.19\times 10^{-10})$. As is well known, one of the multipliers (λ_2) is equal to unity because this is a periodic orbit of an autonomous system. Only λ_1 exceeds unity, so this UPO cannot be stabilized by the original DFC because of the limitation for autonomous systems [11]. Now we attempt to stabilize this UPO by the following half-period DFC:

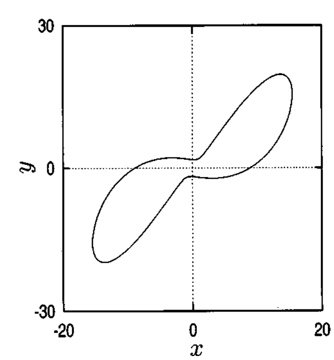


FIG. 7. A symmetric UPO in the Lorenz attractor: An unstable periodic orbit of the Lorenz equation (6) with parameter settings $\sigma=10.0$, $r=28.0$, and $b=8/3$.

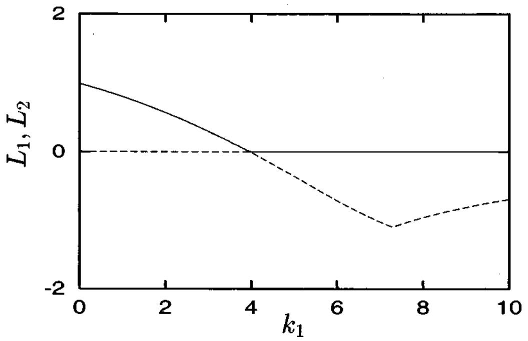


FIG. 8. Characteristic exponents for the symmetric UPO in the Lorenz equation (6) under the half-period DFC. Parameters k_2 and k_3 are fixed to zero. The solid line shows the first characteristic exponent L_1, and the dashed line shows the second characteristic exponent L_2.

$$\dot{x}(t) = -\sigma(x(t)-y(t)) - k_1(x(t-T/2)+x(t)),$$

$$\dot{y}(t) = rx(t) - y(t) - x(t)z(t) - k_2(y(t-T/2)+y(t)), \qquad (15)$$

$$\dot{z}(t) = x(t)y(t) - bz(t) + k_3(z(t-T/2)-z(t)).$$

This is a simplified version of the half-period DFC [Eq. (10)] proposed in Sec. V, with a diagonal gain matrix $K=\mathrm{diag}(k_1,k_2,k_3)$.

Figure 8 shows the first and the second characteristic exponents for the target UPO, with k_1 varied from 0.0 to 10.0 while k_2 and k_3 are fixed to zero. One can see that the first characteristic exponent L_1 decreases as k_1 increases, and becomes zero near $k_1=4.5$. On the other hand, the second characteristic exponent L_2 turns from zero to a negative value at the same value of k_1. That is, the UPO is stabilized by the half-period DFC with some values of k_1 greater than this critical value. An example of the process of stabilizing the UPO with $k_1=5.0$, $k_2=3.0$, and $k_3=0.1$ is shown in Fig. 9. Figure 9(a) is the wave form of z component, and Figs. 9(b), 9(c), and 9(d) are the time evolution of the feedback terms, $x(t-T/2)+x(t)$, $y(t-T/2)+y(t)$, and $z(t-T)-z(t)$, respectively.

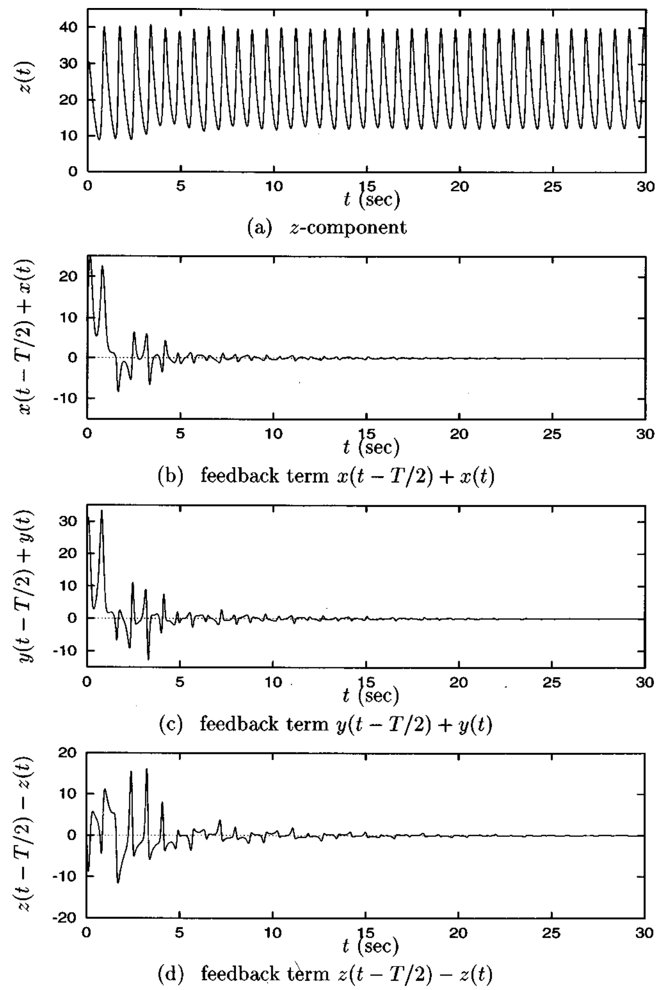


FIG. 9. An example of the stabilization of a UPO of the Lorenz equation (6) by the half-period DFC. The parameter settings are $\sigma=10.0$, $r=28.0$, $b=\frac{8}{3}$, $k_1=5.0$, $k_2=3.0$, and $k_3=0.1$. (a) The wave form of the z component. (b), (c), and (d) show the time evolution of the feedback terms $x(t-T/2)+x(t)$, $y(t-T/2)+y(t)$, and $z(t-T/2)-z(t)$, respectively.

VII. CONCLUSION

In order to overcome the limitation of the DFC, we proposed a half-period delayed feedback control utilizing symmetries of equations. By numerical experiments, we showed that self-symmetrical directly unstable periodic orbits of the Duffing equation and a self-symmetrical UPO of the Lorenz equation could be stabilized by the proposed method.

Since the feedback term vanishes only when the solution is self-symmetric, the coalescence, generation, and vanishing of general T-periodic orbits are possible in the stabilization process by this method. This is the essence of the success of the half-period DFC. It is an interesting problem to clarify the mechanism of the stabilization in detail from such a point of view of bifurcation.

The application of this method is restricted to self-symmetric directly unstable periodic orbits. However, directly unstable periodic orbits are not always self-symmetric, as we described above. To overcome this restriction, we consider the theoretical analysis of the DFC using bifurcation theory, described above, to be very important.

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APPENDIX: A PROPERTY OF SELF-SYMMETRIC SOLUTIONS OF THE DUFFING EQUATION

Here, we prove the following proposition:

Proposition. Every self-symmetric solution of the Duffing equation (3) is directly unstable.

Proof. Let $(x(t),y(t))$ be a self-symmetric periodic solution of the Duffing equation (3). The variational equation of Eq. (3) about this solution is

$$\dot{\xi} = \eta, \quad \dot{\eta} = -\delta\eta - \alpha\xi - 3x(t)^2\xi. \tag{A1}$$

This is a linear periodic equation of the variable $(\xi(t),\eta(t))$. In general, its period is equal to 2π, the period of the reference orbit $(x(t),y(t))$. However, in Eq. (A1), only $3x(t)^2$ on the right hand side is time varying. Since $x(t)$ is self-symmetric, $x(t) = -x(t-\pi)$ by its definition, and thus $x(t)^2 = x(t-\pi)^2$. Hence Eq. (A1) is a linear periodic equation with period π. Therefore by the Floquet theory, even if $(x(t),y(t))$ is inversely unstable, it is so in the sense of a π-periodic solution [i.e., Eq. (A1) has an unstable solution composed of a 2π-periodic solution and an exponential function], but it is directly unstable in the sense of a 2π-periodic solution. As a consequence, a self-symmetric solution is directly unstable.

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